Domain Specific Languages
for Convex Optimization

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On the occasion of Boris Polyak’s 80th birthday
Outline

Constructive convex analysis

Cone representation

Canonicalization

Modeling frameworks

Conclusions
How do you solve a convex problem?

- use someone else’s (‘standard’) solver (LP, QP, SOCP, …)
  - easy, but your problem **must** be in a standard form
  - cost of solver development amortized across many users

- write your own (custom) solver
  - lots of work, but can take advantage of special structure

- transform your problem into a standard form, and use a standard solver
  - extends reach of problems solvable by standard solvers

- **this talk**: methods to formalize and automate last approach
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How can you tell if a problem is convex?

approaches:

▶ use basic definition, first or second order conditions, e.g.,
  \( \nabla^2 f(x) \succeq 0 \)

▶ via convex calculus: construct \( f \) using
  ▶ library of basic functions that are convex
  ▶ calculus rules or transformations that preserve convexity
Convex functions: Basic examples

- $x^p$ ($p \geq 1$ or $p \leq 0$), $-x^p$ ($0 \leq p \leq 1$)
- $e^x$, $-\log x$, $x \log x$
- $a^T x + b$
- $x^T P x$ ($P \succeq 0$)
- $\|x\|$ (any norm)
- $\max(x_1, \ldots, x_n)$
Convex functions: Less basic examples

- $x^T x / y$ ($y > 0$), $x^T Y^{-1} x$ ($Y \succ 0$)
- $\log(e^{x_1} + \cdots + e^{x_n})$
- $- \log \Phi(x)$ ($\Phi$ is Gaussian CDF)
- $\log \det X^{-1}$ ($X \succ 0$)
- $\lambda_{\max}(X)$ ($X = X^T$)
- $f(x) = x[1] + \cdots + x[k]$ (sum of largest $k$ entries)
Calculus rules

▶ **nonnegative scaling:** $f$ convex, $\alpha \geq 0 \implies \alpha f$ convex

▶ **sum:** $f, h$ convex $\implies f + g$ convex

▶ **affine composition:** $f$ convex $\implies f(Ax + b)$ convex

▶ **pointwise maximum:** $f_1, \ldots, f_m$ convex $\implies \max_i f_i(x)$ convex

▶ **partial minimization:** $f(x, y)$ convex $\implies \inf_y f(x, y)$ convex

▶ **composition:** $h$ convex increasing, $f$ convex $\implies h(f(x))$ convex
A general composition rule

\[ h(f_1(x), \ldots, f_k(x)) \] is convex when \( h \) is convex and for each \( i \)

- \( h \) is increasing in argument \( i \), and \( f_i \) is convex, or
- \( h \) is decreasing in argument \( i \), and \( f_i \) is concave, or
- \( f_i \) is affine

- there’s a similar rule for concave compositions
- this one rule subsumes most of the others
- in turn, it can be derived from the partial minimization rule
Constructive convexity verification

- start with function given as expression
- build parse tree for expression
  - leaves are variables or constants/parameters
  - nodes are functions of children, following general rule
- tag each subexpression as convex, concave, affine, constant
  - variation: tag subexpression signs, use for monotonicity
    - e.g., $(\cdot)^2$ is increasing if its argument is nonnegative
- sufficient (but not necessary) for convexity
Example

for $x < 1, y < 1$

\[
\frac{(x - y)^2}{1 - \max(x, y)}
\]

is convex

- (leaves) $x$, $y$, and 1 are affine expressions
- $\max(x, y)$ is convex; $x - y$ is affine
- $1 - \max(x, y)$ is concave
- function $u^2 / \nu$ is convex, monotone decreasing in $\nu$ for $\nu > 0$ hence, convex with $u = x - y$, $\nu = 1 - \max(x, y)$
Example

analyzed by dcp.stanford.edu (Diamond 2014)

Constructive convex analysis
Disciplined convex programming (DCP)

- framework for describing convex optimization problems
- based on constructive convex analysis
- sufficient but not necessary for convexity
- basis for several domain specific languages and tools for convex optimization
Disciplined convex program: Structure

a DCP has

- zero or one **objective**, with form
  - minimize $\{\text{scalar convex expression}\}$ or
  - maximize $\{\text{scalar concave expression}\}$

- zero or more **constraints**, with form
  - $\{\text{convex expression}\} \leq \{\text{concave expression}\}$ or
  - $\{\text{concave expression}\} \geq \{\text{convex expression}\}$ or
  - $\{\text{affine expression}\} = \{\text{affine expression}\}$
Disciplined convex program: Expressions

- expressions formed from
  - variables,
  - constants/parameters,
  - and functions from a library
- library functions have known convexity, monotonicity, and sign properties
- all subexpressions match general composition rule
Disciplined convex program

- A valid DCP is
  - convex-by-construction (cf. posterior convexity analysis)
  - ‘syntactically’ convex (can be checked ‘locally’)

- Convexity depends only on attributes of library functions, and not their meanings
  - E.g., could swap $\sqrt{\cdot}$ and $\sqrt[4]{\cdot}$, or $\exp \cdot$ and $(\cdot)_+$, since their attributes match
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Cone representation

\[(\text{Nesterov, Nemirovsky})\]

cone representation of (convex) function \( f \):

\[ f(x) \text{ is optimal value of cone program} \]

\[
\text{minimize } \quad c^T x + d^T y + e \\
\text{subject to } A \begin{bmatrix} x \\ y \end{bmatrix} = b, \quad \begin{bmatrix} x \\ y \end{bmatrix} \in K
\]

- cone program in \((x, y)\), we but minimize only over \(y\)
- \textit{i.e.}, we define \(f\) by partial minimization of cone program
Examples

- $f(x) = -(xy)^{1/2}$ is optimal value of SDP

  \[
  \begin{align*}
  \text{minimize} & \quad -t \\
  \text{subject to} & \quad \begin{bmatrix} x & t \\ t & y \end{bmatrix} \succeq 0 \\
  \end{align*}
  \]

  with variable $t$

- $f(x) = x[1] + \cdots + x[k]$ is optimal value of LP

  \[
  \begin{align*}
  \text{minimize} & \quad \mathbf{1}^T \lambda - k \nu \\
  \text{subject to} & \quad x + \nu \mathbf{1} = \lambda - \mu \\
  & \quad \lambda \succeq 0, \quad \mu \succeq 0 \\
  \end{align*}
  \]

  with variables $\lambda, \mu, \nu$
SDP representations

Nesterov, Nemirovsky, and others have worked out SDP representations for many functions, e.g.,

- $x^p$, $p \geq 1$ rational
- $-(\det X)^{1/n}$
- $\sum_{i=1}^k \lambda_i(X)$ ($X = X^T$)
- $\|X\| = \sigma_1(X)$ ($X \in \mathbb{R}^{m \times n}$)
- $\|X\|_* = \sum_i \sigma_i(X)$ ($X \in \mathbb{R}^{m \times n}$)

some of these representations are not obvious . . .
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- start with problem in DCP form, with cone representable library functions
- automatically transform to equivalent cone program
Canonicalization: How it’s done

- for each (non-affine) library function \( f(x) \) appearing in parse tree, with cone representation

\[
\begin{align*}
\text{minimize} \quad & c^T x + d^T y + e \\
\text{subject to} \quad & A \begin{bmatrix} x \\ y \end{bmatrix} = b, \quad \begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{K}
\end{align*}
\]

- add new variable \( y \), and constraints above
- replace \( f(x) \) with affine expression \( c^T x + d^T y + e \)
- yields problem with linear equality and cone constraints
- DCP ensures equivalence of resulting cone program
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- constrained least-squares problem with $\ell_1$ regularization

\[
\begin{align*}
\text{minimize} & \quad \|Ax - b\|_2^2 + \gamma \|x\|_1 \\
\text{subject to} & \quad \|x\|_\infty \leq 1
\end{align*}
\]

- variable $x \in \mathbb{R}^n$
- constants/parameters $A$, $b$, $\gamma > 0$
CVX

► developed by M. Grant
► embedded in Matlab; targets multiple cone solvers

► CVX specification for example problem:

```matlab
cvx_begin
    variable x(n)  % declare vector variable
    minimize sum(square(A*x-b,2)) + gamma*norm(x,1)
    subject to norm(x,inf) <= 1
cvx_end
```

► here $A$, $b$, $\gamma$ are **constants**
## Some functions in the CVX library

<table>
<thead>
<tr>
<th>function</th>
<th>meaning</th>
<th>attributes</th>
</tr>
</thead>
<tbody>
<tr>
<td>norm(x, p)</td>
<td>$|x|_p, \ p \geq 1$</td>
<td>cvx</td>
</tr>
<tr>
<td>square(x)</td>
<td>$x^2$</td>
<td>cvx</td>
</tr>
<tr>
<td>square_pos(x)</td>
<td>$(x_+)^2$</td>
<td>cvx, nondecr</td>
</tr>
<tr>
<td>pos(x)</td>
<td>$x_+$</td>
<td>cvx, nondecr</td>
</tr>
<tr>
<td>sum_largest(x,k)</td>
<td>$x[1] + \cdots + x[k]$</td>
<td>cvx, nondecr</td>
</tr>
<tr>
<td>sqrt(x)</td>
<td>$\sqrt{x}, \ x \geq 0$</td>
<td>ccv, nondecr</td>
</tr>
<tr>
<td>inv_pos(x)</td>
<td>$1/x, \ x &gt; 0$</td>
<td>cvx, nonincr</td>
</tr>
<tr>
<td>max(x)</td>
<td>$\max{x_1, \ldots, x_n}$</td>
<td>cvx, nondecr</td>
</tr>
<tr>
<td>quad_over_lin(x,y)</td>
<td>$x^2/y, \ y &gt; 0$</td>
<td>cvx, nonincr in y</td>
</tr>
<tr>
<td>lambda_max(X)</td>
<td>$\lambda_{\max}(X), \ X = X^T$</td>
<td>cvx</td>
</tr>
<tr>
<td>huber(x)</td>
<td>$\begin{cases} x^2, &amp;</td>
<td>x</td>
</tr>
</tbody>
</table>
CVXPY

- developed by S. Diamond
- embedded in Python; targets multiple cone solvers

CVXPY specification for example problem:

```python
from cvxpy import *
x = Variable(n)
cost = sum_squares(A*x-b) + gamma*norm(x,1)
obj = Minimize(cost)
constr = [norm(x,"inf") <= 1]
prob = Problem(obj,constr)
opt_val = prob.solve()
solution = x.value
```
Parameters in CVXPY

- symbolic representations of constants
- can specify sign
- change value of constant without re-parsing problem

- computing a trade-off curve for example problem:

```python
x_values = []
for val in numpy.logspace(-4, 2, num=100):
    gamma.value = val
    prob.solve()
    x_values.append(x.value)
```
## Signed DCP in CVXPY

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<td>$|x|_p$, $p \geq 1$</td>
<td>cvx, nondecr for $x \geq 0$, nonincr for $x \leq 0$</td>
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<tr>
<td>square(x)</td>
<td>$x^2$</td>
<td>cvx, nondecr for $x \geq 0$, nonincr for $x \leq 0$</td>
</tr>
</tbody>
</table>
| huber(x)      | \[
\begin{align*}
   &x^2, & |x| \leq 1 \\
   &2|x| - 1, & |x| > 1
\end{align*}
\] | cvx, nondecr for $x \geq 0$, nonincr for $x \leq 0$ |
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- DCP is a formalization of constructive convex analysis
  - simple method to certify problem as convex
  - basis of several domain specific languages for convex optimization

- modeling frameworks make rapid prototyping easy
Happy Birthday Boris!