Convex Relaxations of Chance Constrained Algebraic Problems

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Chance Optimization Problem

Let

- $q \in \mathbb{R}^m$ be a random variable with probability measure $\mu_q$
- $x \in \mathbb{R}^n$ be a decision variable
- $p_j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, j = 1, 2, \ldots, l$ be polynomials

Solve

$$P_1^* = \max_x \text{Prob}_{\mu_q} \{ q : p_j(x, q) \geq 0, j = 1, 2, \ldots, l \}$$

This is, in general, a hard nonconvex problem
Example: Target Reaching Under Random Uncertainty

Take a polynomial system

\[ x(t + 1) = f[x(t), u(t), \eta(t)] \]

Find a polynomial state feedback law

\[ u(t) = q[x(t)] \]

that maximizes the probability of reaching and remaining in the target.
Example: Explaining Data using Switched Systems

- Find “simple” explanations/models for data
- Find relations in between data collected
- Detect changes in data “behavior”

Examples:

  - gait/activity recognition
  - video shot change detection
  - Gene classification
Other Examples

• Chance Constrained Model Predictive Control

• Fixed Order Probabilistically Robust Controller

• Portfolio Optimization/Risk Minimization

• ...

•
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\]
Previous Work

- Robust Optimization

- Upper Bounds/Approximations of Chance Constraints

- Scenario Approach
Our Objective

Find approximations of this problem that are

i) Convex Problems

ii) Asymptotically Exact

Assumptions

i) \( x \) belongs to the hyper-cube \( \chi = [-1, 1]^n \)

ii) probability measure \( \mu_q \) satisfies \( supp(\mu_q) \subseteq Q = [-1, 1]^m \)
Equivalent Problem in Space of Measures

Define the compact semialgebraic set

\[ \mathcal{K} = \{(x, q) : p_j(x, q) \geq 0, \; j = 1, 2, \ldots, l \} \cap (\chi \times \mathcal{Q}) \]

Then, one can define the equivalent problem

\[ P^*_2 = \max_{\mu, \mu_x} \int d\mu \]

subject to

\[ \mu \leq \mu_x \times \mu_q \]

\[ \text{supp}(\mu_x) \subseteq \chi, \; \text{supp}(\mu) \subseteq \mathcal{K} \]
Equivalent Problem (cont.)

\[ P_1^* = \max_x \text{Prob}_{\mu_q} \{ q : p_j(x, q) \geq 0, j = 1, 2, \ldots, l \} \]

\[ \mathcal{K} = \{ (x, q) : p_j(x, q) \geq 0, j = 1, 2, \ldots, l \} \cap (\mathcal{X} \times \mathcal{Q}) \]
Equivalent Problem

\[ \text{Prob}_{\mu_q} \{ q : p_j(x, q) \geq 0, \ j = 1, 2, \ldots, l \} = \int_{\mathcal{K}} d(\mu_x \times \mu_q) \]

\[ = \max_{\mu} \int d\mu \text{ subject to } \mu \preceq \mu_x \times \mu_q \text{ and } \text{supp}(\mu) \subseteq \mathcal{K} \]
Equivalence Result

**Theorem:** Problem 1 and Problem 2 are equivalent in the following sense

- The optimal values are the same.
- If $\mu_x^*$ be a solution of Problem 2, then, any $x^* \in supp(\mu_x^*)$ is a solution of Problem 1.
- If $x^*$ be a solution of Problem 1, then $\mu_x^* = \delta_{x^*}$ is a solution of Problem 2.
Comments

We have transformed our problem from a hard nonconvex one into a linear program in measure space. Hence

- It is convex (linear in the measures)

- But, it is infinite dimensional

Lets work with the moments of the measures instead
Moments of Measures

Consider a sequence $y$. We say that this sequence is a \textit{moment sequence} if there exists a measure $\mu$ such that

$$y_\alpha = \int x^\alpha d\mu$$

Under some technical conditions, a sequence $y$ is a moment sequence of some measure $\mu$ supported in the set

$$K = \{ x \in \mathbb{R}^n : p_j(x) \geq 0, \; j = 1, 2, \ldots, m \}$$

if the following holds

$$M_d(y) \geq 0, \; M_d(p_jy) \geq 0, \; j = 1, \ldots, m$$

for all integer $d$. 
Two Dimensional Example

\[ M_N(m) = \begin{bmatrix}
M_{0,0}(m) & M_{0,1}(m) & \cdots & M_{0,N}(m) \\
M_{1,0}(m) & M_{1,1}(m) & \cdots & M_{1,N}(m) \\
\vdots & \vdots & \ddots & \vdots \\
M_{N,0}(m) & M_{N,1}(m) & \cdots & M_{N,N}(m)
\end{bmatrix} \]

\[ M_{j,k}(m) = \begin{bmatrix}
m_{j+k,0} & m_{j+k-1,1} & \cdots & m_{j,k} \\
m_{j+k-1,1} & m_{j+k-2,2} & \cdots & m_{j-1,k+1} \\
\vdots & \vdots & \ddots & \vdots \\
m_{k,j} & m_{k-1,j+1} & \cdots & m_{0,j+k}
\end{bmatrix} \]

Moment localization matrix:

\[ M_{N,i}(p_i m)(i,j) = \sum_{\alpha} p_{i,\alpha} m(\beta(i,j) + \alpha) \]
Measure “Coverage”

Recall that given two measures $\mu_1$ and $\mu_2$

$$\mu_1 \preceq \mu_2 \text{ denotes } \mu_1(A) \leq \mu_2(A) \text{ for any measurable set } A$$

Given two measures $\mu_1$ and $\mu_2$ on a compact set $K$, with moment sequences $y_1 = (y_{1\alpha})$ and $y_2 = (y_{2\alpha})$, we have $\mu_1 \preceq \mu_2$ if:

$$M_d(y_2 - y_1) \succeq 0, M_d(p_j(y_2 - y_1)) \succeq 0, \quad j = 1, \ldots, m$$

for every $d \in \mathbb{N}$.
Equivalent Problem in “Moment Space”

Let \( \mathbf{y}, \mathbf{y}_{\times}, \) and \( \hat{\mathbf{y}} \) be the infinite sequence of all moments of measures \( \mu, \mu_{\times}, \) and \( \hat{\mu} = \mu_{\times} \times \mu_{q}, \) respectively.

Solve

\[
P_{3}^{*} = \sup_{\mathbf{y}, \mathbf{y}_{\times}} y_{0}
\]

subject to

\[
M_{\infty}(\mathbf{y}) \geq 0
\]
\[
M_{\infty}(p_{j} \mathbf{y}) \geq 0, \quad j = 1, 2, \ldots, l
\]
\[
M_{\infty}(\mathbf{y}_{\times}) \geq 0
\]
\[
M_{\infty}(\hat{\mathbf{y}} - \mathbf{y}) \geq 0
\]
Finite Dimensional Approximation

Solve

\[ P_4^i = \sup_{y, y_x} y_0 \]

subject to

\[ M_i(y) \succeq 0 \]

\[ M_{i-r_j}(p_j y) \succeq 0, \; j = 1, 2, \ldots, l \]

\[ M_i(y_x) \succeq 0 \]

\[ M_i(\hat{y} - y) \succeq 0 \]

**Theorem:** Optimal value of problem \( P_4^i \) converges to optimal value of problem \( P_3 \) as \( i \to \infty \).
Comments on Implementation

Solve

\[
\inf_{y, y_x} \| M_i(y_x) \|_* \\
y_0 \geq \gamma \\
M_i(y) \succeq 0 \\
M_{i-r_j}(p_j y) \succeq 0 \quad j = 1, 2, \ldots, l \\
M_i(y_x) \succeq 0 \\
M_i(\hat{y} - y) \succeq 0
\]
Example 1

$$\max_x \text{Prob}_{\mu_q} \left\{ q : \ p(x, q) = -\frac{1}{2} q (q^2 + (x - \frac{1}{2})^2 ) + (q^4 + q^2 (x - \frac{1}{2})^2 + (x - \frac{1}{2})^4 ) \geq 0 \right\}$$

The uncertain parameter $q : \mu_q = U[-1, 1]$
Example 1: Moment Vectors

Moment vector of measure $\mu$

$$y = [y_{00}, y_{10}, y_{01}, y_{20}, y_{11}, y_{02}, y_{30}, y_{21}, y_{12}, y_{03}, y_{40}, y_{31}, y_{22}, y_{13}, y_{04}]$$

Moment vector of measure $\mu_x$

$$y_x = [1, y_{x1}, y_{x2}, y_{x3}, y_{x4}]$$

Moment vector of measure $\mu_q$

$$y_q = [1, y_{q1}, y_{q2}, y_{q3}, y_{q4}] = [1, 0, \frac{1}{3}, 0, \frac{1}{5}, 0]$$

Moment vector of measure $\hat{\mu} = \mu_x \times \mu_q$

$$y_x y_q = [1|y_{x1}, y_{q1}|y_{x2}, y_{x1}y_{q1}, y_{q2}|y_{x3}, y_{x2}y_{q1}, y_{x1}y_{q2}, y_{q3}|y_{x4}, y_{x3}y_{q1}, y_{x2}y_{q2}, y_{x1}y_{q3}, y_{q4}]$$

$$= [1|y_{x1}, 0|y_{x2}, 0, \frac{1}{3}1|y_{x3}, 0, \frac{1}{3}y_{x1}, 0|y_{x4}, 0, \frac{1}{3}y_{x2}, 0, \frac{1}{5}1]$$
Example 1: Optimization Problem

\[ \min_{\gamma, y_1, y_2} \| M_4(y) \|_* = \left\| \begin{pmatrix} 1 & y_1 & y_2 \\ y_1 & y_2 & y_3 \\ y_2 & y_3 & y_4 \end{pmatrix} \right\|_* \]

\[ y_0 \geq \gamma \]

\[ M_4(y) \succeq 0 \Rightarrow \begin{pmatrix} y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{pmatrix} \succeq 0 \]

\[ M_4(y_{x,y_{aq}}) - M_4(y) \succeq 0 \Rightarrow \begin{pmatrix} 1 & y_{x1} & 0 & y_{x2} & 0 & 1/3 \\ y_{x1} & y_{x2} & 0 & y_{x3} & 0 & 1/3y_{x1} \\ 0 & 0 & 1/3 & 0 & 1/3y_{x1} & 0 \\ y_{x2} & y_{x3} & 0 & y_{x4} & 0 & 1/3y_{x2} \\ 0 & 0 & 1/3y_{x1} & 0 & 1/3y_{x2} & 0 \\ 1/3 & 1/3y_{x1} & 0 & 1/3y_{x2} & 0 & 2/5 \end{pmatrix} - M_4(y) \succeq 0 \]

\[ M_4(p_y) \succeq 0 \Rightarrow -y_{04} + \frac{1}{2}y_{03} - y_{22} + y_{12} - \frac{1}{4}y_{02} + \frac{1}{2}y_{21} - \frac{1}{2}y_{11} + \frac{1}{8}y_{01} - y_{40} + 2y_{30} - \frac{3}{2}y_{20} + \frac{1}{2}y_{10} - \frac{1}{16} \succeq 0 \]
Example 1: Results

Obtained Moments

\[ y = [0.58, 0.49, -0.15, 0, 0, 0.18, 0, 0, 0, -0.07, 0, 0, 0, 0] \]

Eigenvalues of \( M_4(x) \): \([0, 0, 0, 0, 0.1, 1]\): \( \text{Rank} (M_4(x)) \approx 1 \): \( \mu_x \approx \text{Dirac measure} \)
Optimal \( x^* \): \( y_{x_1} = 0.499 \)
Optimal Probability: \( y_{00} = 0.58 \)
Example 2

\[
\sup_{x \in \mathbb{R}^5} \mu_q \left( \{ q \in \mathbb{R}^5 : \mathcal{P}(x, q) \geq 0 \} \right)
\]

where

\[
\mathcal{P}(x, q) = 0.185 + 0.5x_1 - 0.5x_2 + x_3 - x_4 + 0.5q_1 - 0.5q_2 + q_3 - q_4 - x_1^2
\]
\[
2x_1q_1 - x_2^2 - 2x_2q_2 - x_3^2 - 2x_3q_3 - x_4^2 - 2x_4q_4 - x_5^2 + 2x_5q_5
\]
\[
q_1^2 - q_2^2 - q_3^2 - q_4^2 - q_5^2,
\]

\[q_1 \sim U[-1, 0], \ q_2 \sim U[0, 1], \ q_3 \sim U[-0.5, 1], \ q_4 \sim U[-1, 0.5], \ q_5 \sim U[0, 1]\]

Optimum (Monte Carlo):

\[x_1^* = 0.75, \ x_2^* = -0.75, \ x_3^* = 0.25, \ x_4^* = -0.25, \ x_5^* = 0.5\]

\[P^* = 0.75\]
## Example 2: Numerical Results

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Concluding Remarks

In this work

• Proposed convex relaxations for a very general class of chance optimization problems

• It is asymptotically exact

Future work

• But it is computationally expensive – Need better "optimization tools"

• Further work is needed in exploiting structure
Thank you

Questions?